Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 292 (2006) 726–741
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Three-dimensional free vibration of thick circular plates on Pasternak foundation

D. Zhou[a,*], S.H. Lo[b], F.T.K. Au[b], Y.K. Cheung[b]

[a]*College of Civil Engineering, Nanjing University of Technology, Nanjing 210009, People's Republic of China*
[b]*Department of Civil Engineering, The University of Hong Kong, Pokfulam Road Hong Kong, People's Republic of China*

Received 16 March 2005; received in revised form 13 July 2005; accepted 26 August 2005
Available online 5 December 2005

Abstract

This paper describes a study of the three-dimensional vibration characteristics of thick circular plates resting on elastic foundation. The analysis is based on the three-dimensional small-strain, linear and exact elasticity theory. The foundation is described by the Pasternak model. The Ritz method is used to derive the frequency equation of the plate-foundation system by augmenting the strain energy of the plate with the elastic potential energy of the foundation. A set of Chebyshev polynomials multiplied by a boundary function is adopted as the admissible functions of the displacement components in each direction. For plates with free edges, the effect of foundation medium beyond the edge of the plate has been considered by introducing the generalized shearing force concept in the analysis. The convergence and comparison studies demonstrate the correctness and accuracy of the present method. It is shown that the present method has rapid convergent rate, stable numerical operation and very high accuracy. Parametric investigations on the dynamic behavior of thick circular plates resting on elastic foundation have been carried out with respect to various thickness–radius ratios, foundation stiffness parameters and boundary conditions. Results known for the first time have been reported and discussed in detail. Some significant conclusions have been drawn.

1. Introduction

Circular plates [1] have wide applications in various fields of engineering. Sometimes, they are used as structural elements resting on foundations, such as building footings, base of machines, etc. The Pasternak model [2] or the two-parameter model is widely adopted to describe the mechanical behavior of foundations, and the well-known Winkler model [3] is one of its special cases.

A lot of research work about circular plates on elastic foundation can be found in the literature, all of which is, however, based on the two-dimensional plate theories. Kamal and Durvasula [4] studied the static bending of a circular plate resting on Pasternak foundation and subjected to uniformly distributed load by using the

*Corresponding author.
E-mail address: dingzhou57@yahoo.com (D. Zhou).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.08.028

Lanczos method. Zheng and Zhou [5] studied the large deflection of a circular plate resting on Winkler foundation and subjected to a concentrated load at the center by means of a step-by-step iterative technique. Celep [6] studied the contact region of circular plates resting on unilateral Winkler foundation and subjected to arbitrary loads by using the minimization of total potential energy. Khathlan [7] studied the effect of large deformation on the contact of a circular plate resting on unilateral Winkler foundation by using the iterative-incremental algorithm. Nassar [8] studied the dynamic response of circular plates on a linear, viscoelastic half-space by using the Laplace transform. Elishakoff and Tang [9] studied the buckling of a circular plate on Winkler foundation by using the Rayleigh's method. Voyiadjis and Kattan [10] presented a two-dimensional elastostatic theory for thick circular plates on Winkler foundation to examine the effect of transverse normal strain incorporating the effects of the transverse shear and normal stress. Ghosh [11] studied the free and forced vibration of circular plates on Winkler foundation by an exact analytical method. Wang et al. [12] derived the exact axisymmetric buckling solutions of Reddy circular plates on Pasternak foundation in terms of the corresponding Kirchhoff solutions. Omurtag et al. [13] used the mixed finite element method to study the free vibration of thin plates on Pasternak foundation. Later Eratli and Akoz [14] extended this method to the free vibration analysis of Reissner plates on Pasternak foundation. Buczkowski and Torbacki [15] studied the static bending of Mindlin plates on Pasternak foundation by using the finite element method. Dumir [16] studied the geometrically nonlinear response of a circular plate resting on Pasternak foundation and subjected to a uniformly distributed static or dynamic load by using the point-collocation method for the spatial discretization and the step-increment method for time. Celep and Turhan [17] studied the response of a circular plate resting on unilateral Winkler foundation and subjected to a concentrated dynamic load at its center by using the Galerkin method. However, little has been done on the vibration of circular plates with free edges on Pasternak foundation.

Two-dimensional theories of plates reduce the dimension of problem from three to two by introducing some assumptions in mathematical modeling. This results in relatively simple formulation and solution. However, these simplifications inherently bring about errors and sometimes loss of some modes. Some researchers have studied the three-dimensional vibration characteristics of circular plates by different methods including the analytical method [18–21] and Ritz method [22–24]. Again, little has been done on the three-dimensional vibration of circular plates on elastic foundation. For a thin plate on an elastic foundation, the common assumption made in most two-dimensional theories that the foundation forces are acting on the median surface of the plate does not result in significant errors. However, for a thick plate such an assumption is unreasonable because the foundation stresses are actually acting on the lower surface of the plate, and therefore the effect on the deformations of the upper and lower surfaces is obviously different [25]. In such a case, the three-dimensional elasticity theory not only provides realistic results but also allows overall physical insights [26].

This paper describes an investigation of the three-dimensional free vibration of thick circular plates resting on Pasternak foundation. The analysis is based on the small-strain, linear and exact elasticity theory. Using the Chebyshev polynomial series as the admissible functions, the eigenvalue equation is derived by using the Ritz method. For plates with free edges, the generalized shearing force concept is introduced to describe the effect of foundation medium beyond the plate. Convergence and comparison studies verify the accuracy of the present method. Some results known for the first time are reported.

2. Formulation

Consider a homogeneous isotropic circular plate with a radius R and a thickness h, which is resting on an elastic foundation as shown in Fig. 1. A cylindrical coordinate system (r, θ, z) with the origin o at the plate center is used to describe the plate displacement. The Pasternak model is used to describe the reaction of the foundation on the plate. The displacement components of the plate in the r, θ and z directions are u, v and w, respectively.

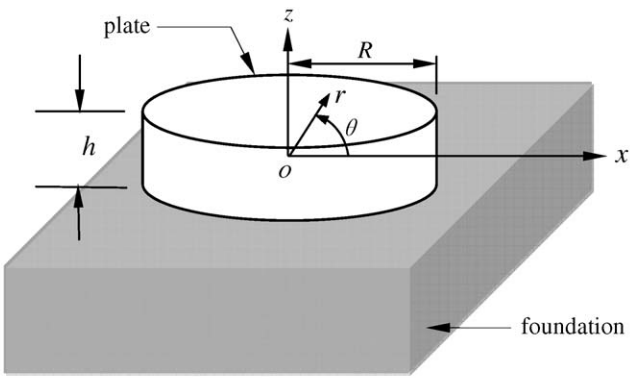

Fig. 1. A thick circular plate on elastic foundation.

Based on the small-strain, linear and exact elasticity theory, the total elastic strain energy V of the plate-foundation system is given by

$$V = \frac{G}{2}\int_0^R \int_0^{2\pi} \int_{-h/2}^{h/2} \left[\frac{2v}{1-2v}(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz})^2 + 2(\varepsilon_{rr}^2+\varepsilon_{\theta\theta}^2+\varepsilon_{zz}^2) + \varepsilon_{r\theta}^2+\varepsilon_{rz}^2+\varepsilon_{\theta z}^2 \right] r\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r$$
$$+\frac{k_1}{2}\int_0^R \int_0^{2\pi} (w|_{z=-h/2})^2 r\,\mathrm{d}\theta\,\mathrm{d}r + \frac{k_2}{2}\int_0^R \int_0^{2\pi} \left[\left(\frac{\partial w}{\partial r}\bigg|_{z=-h/2} \right)^2 + \left(\frac{\partial w}{r\,\partial\theta}\bigg|_{z=-h/2} \right)^2 \right] r\,\mathrm{d}\theta\,\mathrm{d}r, \tag{1}$$

where G is the shear modulus, v is the Poisson's ratio, and k_1 is the Winkler foundation stiffness while k_2 is the shear stiffness of the elastic foundation. The strain components ε_{ij} $(i,j = r,\theta,z)$ are defined as follows:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{\partial v}{r\,\partial\theta}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z},$$
$$\varepsilon_{r\theta} = \frac{\partial u}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad \varepsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{\partial w}{r\,\partial\theta}. \tag{2}$$

The kinetic energy T of the plate can be given as

$$T = \frac{\rho}{2}\int_0^R \int_0^{2\pi} \int_{-h/2}^{h/2} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] r\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r, \tag{3}$$

where ρ is the mass density per unit volume.

For simplicity and convenience in the mathematical formulation, the following dimensionless parameters are introduced:

$$\bar{r} = \frac{2r}{R} - 1, \quad \bar{\theta} = \theta, \quad \bar{z} = \frac{2z}{h}. \tag{4}$$

For free vibration analysis, the displacement components of the plate can be expressed in terms of the displacement amplitude functions as follows:

$$u(r,\theta,z,t) = U(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}, \quad v(r,\theta,z,t) = V(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}, \quad w(r,\theta,z,t) = W(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}, \tag{5}$$

where ω denotes the frequency of the plate and $\mathrm{i} = \sqrt{-1}$.

Considering the circumferential symmetry of the circular plate about the coordinate $\bar{\theta}$, the displacement amplitude functions can be expressed by

$$U(\bar{r},\bar{\theta},\bar{z}) = \bar{U}(\bar{r},\bar{z})\cos(s\bar{\theta}), \quad V(\bar{r},\bar{\theta},\bar{z}) = \bar{V}(\bar{r},\bar{z})\sin(s\bar{\theta}), \quad W(\bar{r},\bar{\theta},\bar{z}) = \bar{W}(\bar{r},\bar{z})\cos(s\bar{\theta}), \tag{6}$$

where s is the circumferential wavenumber, which should be taken to be an integer (namely $s = 0, 1, 2, \ldots, \infty$) to ensure the periodicity in the $\bar{\theta}$ direction. It is obvious that $s = 0$ means axisymmetric vibration. In such a case, $U(\bar{r},\bar{\theta},\bar{z})=\bar{U}(\bar{r},\bar{z})$, $V(\bar{r},\bar{\theta},\bar{z})=0$, and $W(\bar{r},\bar{\theta},\bar{z})=\bar{W}(\bar{r},\bar{z})$. Rotating the symmetry axes by $\pi/2$, another set of free vibration modes can be obtained, which corresponds to an interchange of $\cos(s\theta)$ and $\sin(s\theta)$ in Eq. (6). However, in such a case, $s=0$ means $U(\bar{r},\bar{\theta},\bar{z})=0$, $V(\bar{r},\bar{\theta},\bar{z})=\bar{V}(\bar{r},\bar{z})$, and $W(\bar{r},\bar{\theta},\bar{z})=0$, representing torsional vibration.

Substituting Eqs. (4)–(6) into Eqs. (1)–(3) gives the maximum potential energy $V_{\max}$ and kinetic energy $T_{\max}$ of the plate, respectively, as

$$\begin{aligned}V_{\max}=&\frac{Gh}{4}\int_{-1}^{1}\int_{-1}^{1}\left[\frac{2v}{1-2v}\Gamma_1(\bar{\varepsilon}_{\bar{r}\bar{r}}+\bar{\varepsilon}_{\bar{\theta}\bar{\theta}}+\bar{\varepsilon}_{\bar{z}\bar{z}})^2+2\Gamma_1(\bar{\varepsilon}_{\bar{r}\bar{r}}^2+\bar{\varepsilon}_{\bar{\theta}\bar{\theta}}^2+\bar{\varepsilon}_{\bar{z}\bar{z}}^2)+\Gamma_1\bar{\varepsilon}_{\bar{r}\bar{z}}^2+\Gamma_2(\bar{\varepsilon}_{\bar{r}\bar{\theta}}^2+\bar{\varepsilon}_{\bar{\theta}\bar{z}}^2)\right]\\&\times(\bar{r}+1)\mathrm{d}\bar{z}\,\mathrm{d}\bar{r}+\frac{Gh}{2}\left(\frac{R}{h}\right)\int_{-1}^{1}\left[\frac{\Phi_1}{4}\Gamma_1(\bar{W}|_{\bar{z}=-1})^2+\Phi_2\Gamma_1\left(\left.\frac{\partial\bar{W}}{\partial\bar{r}}\right|_{\bar{z}=-1}\right)^2\right.\\&\left.+\Phi_2 s^2\Gamma_2\frac{1}{(\bar{r}+1)^2}(\bar{W}|_{\bar{z}=-1})^2\right](\bar{r}+1)\mathrm{d}\bar{r},\\T_{\max}=&\frac{\rho R^2h}{16}\omega^2\int_{-1}^{1}\int_{-1}^{1}(\Gamma_1\bar{U}^2+\Gamma_2\bar{V}^2+\Gamma_1\bar{W}^2)(\bar{r}+1)\mathrm{d}\bar{z}\,\mathrm{d}\bar{r}\end{aligned}\tag{7}$$

in terms of the strains

$$\begin{aligned}&\bar{\varepsilon}_{\bar{r}\bar{r}}=\frac{\partial\bar{U}}{\partial\bar{r}},\quad\bar{\varepsilon}_{\bar{\theta}\bar{\theta}}=\frac{\bar{U}}{\bar{r}+1}+\frac{s\bar{V}}{\bar{r}+1},\quad\bar{\varepsilon}_{\bar{z}\bar{z}}=\frac{\partial\bar{W}}{\gamma\,\partial\bar{z}},\\&\bar{\varepsilon}_{\bar{r}\bar{\theta}}=-\frac{s\bar{U}}{\bar{r}+1}+\frac{\partial\bar{V}}{\partial\bar{r}}-\frac{\bar{V}}{\bar{r}+1},\quad\bar{\varepsilon}_{\bar{r}\bar{z}}=\frac{\partial\bar{U}}{\gamma\,\partial\bar{z}}+\frac{\partial\bar{W}}{\partial\bar{r}},\quad\bar{\varepsilon}_{\bar{\theta}\bar{z}}=\frac{\partial\bar{V}}{\gamma\,\partial\bar{z}}-\frac{s\bar{W}}{\bar{r}+1}\end{aligned}\tag{8}$$

and in terms of the parameters

$$\begin{aligned}&\Phi_1=k_1R/G,\quad\Phi_2=k_2/(GR),\\&\Gamma_1=\int_0^{2\pi}\cos^2(s\theta)\mathrm{d}\theta=\begin{cases}2\pi & \text{if}\quad s=0,\\ \pi & \text{if}\quad s>0,\end{cases}\\&\Gamma_2=\int_0^{2\pi}\sin^2(s\theta)\mathrm{d}\theta=\begin{cases}0 & \text{if}\quad s=0,\\ \pi & \text{if}\quad s>0,\end{cases}\qquad\gamma=h/R.\end{aligned}\tag{9}$$

It is assumed that each of the displacement amplitude functions are separable in variable and can be written in the form of double series of Chebyshev polynomials multiplied by boundary functions as follows:

$$\begin{aligned}\bar{U}(\bar{r},\bar{z})&=F_u(\bar{r})\sum_{i=1}^{I}\sum_{j=1}^{J}A_{ij}P_i(\bar{r})P_i(\bar{z}),\\\bar{V}(\bar{r},\bar{z})&=F_v(\bar{r})\sum_{k=1}^{K}\sum_{l=1}^{L}B_{kl}P_k(\bar{r})P_l(\bar{z}),\\\bar{W}(\bar{r},\bar{z})&=F_w(\bar{r})\sum_{m=1}^{M}\sum_{n=1}^{N}C_{mn}P_m(\bar{r})P_n(\bar{z}),\end{aligned}\tag{10}$$

where I, J, K, L, M and N are the truncated orders of the Chebyshev polynomial series and A_{ij}, B_{kl} and C_{mn} are coefficients yet to be determined. $P_i(\chi)$ $(i=1,2,3,\ldots;\ \chi=\bar{r},\bar{z})$ is the ith one-dimensional Chebyshev polynomial which can be written exactly in terms of cosine functions as follows:

$$P_i(\chi)=\cos[(i-1)\arccos(\chi)]\quad(i=1,2,3,\ldots).\tag{11}$$

The boundary functions $F_u(\bar{r})$, $F_v(\bar{r})$ and $F_w(\bar{r})$ should enable the displacement components u, v and w to satisfy the geometric boundary conditions of the plate. However, the stress boundary conditions need not be satisfied in advance. The boundary functions corresponding to common boundary conditions are given in Table 1.

Table 1
Boundary functions for different boundary conditions

Boundary conditions	$F_u(\bar{r})$	$F_v(\bar{r})$	$F_w(\bar{r})$
Clamped	$1+\bar{r}$	$1+\bar{r}$	$1+\bar{r}$
Completely free	1	1	1
Hard simply supported	1	$1+\bar{r}$	$1+\bar{r}$
Sliding	$1+\bar{r}$	1	1
Soft simply supported	1	1	$1+\bar{r}$

It should be noted that Chebyshev polynomial series $P_i(\chi)$ $(i = 1, 2, 3, \ldots)$ is a set of complete and orthogonal series in the interval [−1,1]. This therefore ensures that the double series $P_i(\bar{r})P_j(\bar{z})$ $(i, j = 1, 2, 3, \ldots)$ is also a complete and orthogonal set in the plate region. The excellent properties of Chebyshev polynomial series in the approximation of functions are well known [27]. Therefore, more rapid convergence and better stability in numerical operation than other polynomial series can be expected [26].

It should be mentioned that Eq. (1) is only suitable for plates with zero vertical displacements along the plate edge at $r = R$ and $z = -h/2$, e.g. clamped or simply supported plates; otherwise a generalized shearing force should be considered in the analysis [28]. Beyond the edge of the plate, the differential equation of the foundation is

$$k_2\left(\frac{\partial^2 w_e}{\partial r^2}+\frac{1}{r}\frac{\partial w_e}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_e}{\partial \theta^2}\right)-k_1 w_e=0, \tag{12}$$

where w_e is the normal displacement of the soil medium. The general solution of the above equation can be easily obtained as

$$w_e(r,\theta,t)=\mathrm{e}^{\mathrm{i}\omega t}W_e(r,\theta)=\mathrm{e}^{\mathrm{i}\omega t}[A_{1s}I_s(\alpha r)+B_{1s}K_s(\alpha r)]\cos(s\theta), \tag{13}$$

where A_{1s} and B_{1s} are unknown coefficients, $\alpha=\sqrt{k_1/k_2}$, and $I_s(\alpha r)$ and $K_s(\alpha r)$ are the first and second kinds of modified Bessel functions of order s, respectively. Rotating the symmetry axes by $\pi/2$, another set of free vibration modes can be obtained, which corresponds to an interchange of $\cos(s\theta)$ and $\sin(s\theta)$ in the above equation.

At infinite values of r, the displacements of the soil medium should be zero. It is clear that $I_s(\alpha r)$ is a monotonically increasing function with respect to r. Therefore, one has

$$W_e(r,\theta)=B_{1s}K_s(\alpha r)\cos(s\theta)=\bar{W}_e(r)\cos(s\theta). \tag{14}$$

According to the continuity of vertical foundation displacements at the plate edge, one has

$$W|_{\bar{r}=1,\bar{z}=-1}=W_e|_{r=R}. \tag{15}$$

Substituting Eqs. (6) and (14) into the above equation, one has

$$B_{1s}=\frac{1}{K_s(\alpha R)}\bar{W}|_{\bar{r}=1,\bar{z}=-1}. \tag{16}$$

As pointed out by Selvadurai [28], the generalized shearing force due to the shearing stresses in the soil medium extending beyond the boundary of the foundation can be given as

$$S_r=-k_2\frac{\partial w_e}{\partial r}\bigg|_{r=R}=k_3G\bar{W}|_{\bar{r}=1,\bar{z}=-1}\cos(s\bar{\theta}),\qquad k_3=-\frac{\dot{K}_s(\alpha R)}{K_s(\alpha R)}\sqrt{\Phi_1\Phi_2}, \tag{17}$$

where $\dot{K}_s(\alpha R)=[\mathrm{d}K_s(x)/\mathrm{d}x]_{x=\alpha R}$ and $\alpha R=\sqrt{\Phi_1/\Phi_2}$. Observing the above equation, one can find that the generalized shearing force acts as translational springs distributed along the plate edge at $r = R$ and $z = -h/2$ while the equivalent spring stiffness per unit length is just equal to k_3G.

In such a case, a complimentary potential energy $V^b_{\max}$ provided by the boundary spring $k_3 G$ should be added into the potential energy of the plate-foundation system

$$V^b_{\max} = \frac{1}{2} k_3 G \int_0^{2\pi} [\bar{W}|_{\bar{r}=1,\bar{z}=-1} \cos(s\theta)]^2 R \, \mathrm{d}\theta = \frac{Gh}{2} \Gamma_1 \Phi_3 \bar{W}^2|_{\bar{r}=1,\bar{z}=-1}, \tag{18}$$

where $\Phi_3 = k_3/\gamma$.

The energy functional of the plate is defined as follows:

$$\Pi = V_{\max} - T_{\max} \tag{19a}$$

for a plate having zero vertical displacements at the plate edge at $\bar{r} = 1$ and $\bar{z} = -1$, and

$$\Pi = V_{\max} + V^b_{\max} - T_{\max} \tag{19b}$$

for a plate having vertical displacements at the plate edge at $\bar{r} = 1$ and $\bar{z} = -1$.

Minimizing Π with respect to the coefficients, i.e.

$$\frac{\partial \Pi}{\partial A_{ij}} = 0, \quad \frac{\partial \Pi}{\partial B_{kl}} = 0, \quad \frac{\partial \Pi}{\partial C_{mn}} = 0 \tag{20}$$

leads to the following eigenvalue equation:

$$\left[\begin{pmatrix} [K^{uu}] & [K^{uv}] & [K^{uw}] \\ & K^{vv} & [K^{vw}] \\ Sym & & [K^{ww}] \end{pmatrix} - \Omega^2 \begin{pmatrix} [M^{uu}] & [0] & [0] \\ & [M^{vv}] & [0] \\ Sym & & [M^{ww}] \end{pmatrix} \right] \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}, \quad s \geqslant 1. \tag{21}$$

In particular, it can be simplified for axisymmetric mode as

$$\left[\begin{pmatrix} [K^{uu}] & [K^{uw}] \\ Sym & [K^{ww}] \end{pmatrix} - \Omega^2 \begin{pmatrix} [M^{uu}] & [0] \\ Sym & [M^{ww}] \end{pmatrix} \right] \begin{Bmatrix} \{A\} \\ \{C\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix}, \quad s = 0 \tag{22}$$

and for torsional mode as

$$([K^{vv}] - \Omega^2 [M^{vv}])\{B\} = \{0\}, \quad s = 0 \tag{23}$$

in which $\Omega = \omega R \sqrt{\rho/G}$, and $[K^{ij}]$ and $[M^{ij}]$ $(i, j = u, v, w)$ are the stiffness sub-matrices and the diagonal mass sub-matrices, respectively. A detailed description of the elements in the eigenvalue equation is given in the Appendix.

A non-trivial solution is obtained by setting the determinant of the coefficient matrix of Eqs. (21)–(23) to zero, respectively. The roots of the determinant are the square of the eigenvalue or dimensionless natural frequency Ω. The eigenfunctions, or the mode shapes, corresponding to the respective natural frequencies are determined by back-substitution of the eigenvalues, one by one, in the usual manner.

3. Convergence and comparison study

All natural frequencies obtained from the Ritz method are upper bounds of the exact ones, and therefore convergence should be monotonic from above as the number of terms of the admissible functions increases. The convergence study was carried out for circular plates with Poisson's ratio $v = 0.3$ and resting on a Pasternak foundation with stiffness parameters $\Phi_1 = 100$ and $\Phi_2 = 10$. For simplicity, equal numbers of terms of the Chebyshev polynomial series were used for the displacement amplitude functions U, V and W, i.e. $I = K = M$ and $J = L = N$, although in most cases, using unequal numbers of terms may result in more rapid convergence with less computational cost. It is obvious that the torsional vibration of the plate is independent of the foundation described by the Pasternak model, which has been well reported in the literature [26]. Therefore in the following analysis, attention will concentrate on the axisymmetric vibration ($s = 0$) and the circumferential vibrations ($s > 0$). The convergence rates of the first six frequency parameters for $s = 0, 1, 2, 3$ with respect to four different combinations of number of terms of Chebyshev polynomials have been checked for circular plates with three kinds of boundary conditions and different thickness ratios. Table 2 gives the

Table 2
Convergence of the first six frequency parameters for a completely free circular plate with $\nu = 0.3$ and $\gamma = 0.3$, resting on Pasternak foundation with $\Phi_1 = 100$ and $\Phi_2 = 10$

$I \times J$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$s = 0$						
6 × 3	3.397	7.406	7.784	8.926	9.864	10.742
8 × 4	3.397	7.376	7.587	8.863	9.634	10.610
10 × 5	3.397	7.376	7.583	8.762	9.640	10.601
15 × 8	3.397	7.376	7.583	8.762	9.640	10.601
$s = 1$						
6 × 3	2.726	5.652	6.758	7.803	8.451	9.265
8 × 4	2.726	5.647	6.742	7.655	8.338	9.102
10 × 5	2.726	5.647	6.742	7.652	8.337	9.099
15 × 8	2.726	5.647	6.742	7.652	8.337	9.099
$s = 2$						
6 × 3	2.345	4.203	6.969	7.817	8.344	9.466
8 × 4	2.345	4.203	6.946	7.683	8.242	9.254
10 × 5	2.345	4.203	6.946	7.681	8.241	9.252
15 × 8	2.345	4.203	6.946	7.681	8.241	9.251
$s = 3$						
6 × 3	3.599	5.693	7.698	8.271	9.474	10.564
8 × 4	3.599	5.688	7.625	8.130	9.400	10.141
10 × 5	3.599	5.688	7.623	8.128	9.399	10.133
15 × 8	3.599	5.688	7.523	8.128	9.399	10.132

Table 3
Convergence of the first six frequency parameters for a clamped circular plate with $\nu = 0.3$ and $\gamma = 0.2$, resting on Pasternak foundation with $\Phi_1 = 100$ and $\Phi_2 = 10$

$I \times J$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$s = 0$						
8 × 3	6.416	10.644	12.725	13.036	14.202	14.728
10 × 4	6.412	10.611	12.640	12.906	14.039	14.535
15 × 5	6.411	10.609	12.638	12.903	14.034	14.526
20 × 7	6.410	10.608	12.638	12.903	14.033	14.525
$s = 1$						
8 × 3	3.331	5.370	8.408	8.798	11.651	11.857
10 × 4	3.330	5.370	8.405	8.790	11.630	11.815
15 × 5	3.329	5.370	8.404	8.789	11.629	11.814
20 × 7	3.329	5.370	8.404	8.788	11.629	11.813
$s = 2$						
8 × 3	5.176	6.890	9.817	10.497	12.585	13.243
10 × 4	5.173	6.889	9.813	10.472	12.506	13.181
15 × 5	5.172	6.889	9.812	10.470	12.505	13.180
20 × 7	5.172	6.889	9.812	10.469	12.504	13.180
$s = 3$						
8 × 3	6.717	8.428	11.073	11.746	13.303	14.665
10 × 4	6.713	8.424	11.054	11.705	13.186	14.466
15 × 5	6.712	8.423	11.053	11.703	13.184	14.458
20 × 7	6.711	8.423	11.053	11.702	13.184	14.456

Table 4
Convergence of the first six frequency parameters for a soft simply supported circular plate with $v = 0.3$ and $\gamma = 0.1$, resting on Pasternak foundation with $\Phi_1 = 100$ and $\Phi_2 = 10$

$I \times J$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$s = 0$						
10 × 2	3.460	9.028	14.137	18.836	23.002	25.891
15 × 3	3.459	9.015	14.049	18.411	21.578	23.696
20 × 4	3.459	9.015	14.045	18.388	21.505	23.578
25 × 6	3.459	9.015	14.045	18.387	21.504	23.576
$s = 1$						
10 × 2	2.741	5.948	6.836	9.948	11.510	13.188
15 × 3	2.739	5.945	6.835	9.947	11.477	13.178
20 × 4	2.739	5.945	6.835	9.947	11.476	13.178
25 × 6	2.739	5.945	6.835	9.947	11.476	13.178
$s = 2$						
10 × 2	2.346	4.259	7.626	8.754	11.414	13.698
15 × 3	2.346	4.256	7.621	8.749	11.412	13.628
20 × 4	2.346	4.255	7.621	8.749	11.411	13.625
25 × 6	2.346	4.255	7.621	8.749	11.411	13.625
$s = 3$						
10 × 2	3.604	5.855	9.029	10.676	12.870	15.576
15 × 3	3.603	5.849	9.022	10.661	12.859	15.467
20 × 4	3.603	5.848	9.021	10.660	12.859	15.462
25 × 6	3.603	5.848	9.021	10.660	12.859	15.462

Table 5
Comparison of the first five frequency parameters λ for a clamped thin circular plate ($\gamma = 0.001$) on a Winkler foundation ($\Phi_1 = 10^{-6}$, $\Phi_2 = 0$) with those from the classical plate theory

s	Methods	λ_1	λ_2	λ_3	λ_4	λ_5
0	Present	65.605	76.042	110.20	170.98	255.42
	Classical	65.608	76.038	110.18	170.94	255.37
1	Present	68.207	88.893	136.48	209.38	304.80
	Classical	68.205	88.877	136.45	209.34	304.74
2	Present	73.602	106.57	166.95	251.28	357.34
	Classical	73.598	106.55	166.91	251.21	357.31
3	Present	82.496	128.58	201.08	296.42	412.93
	Classical	82.493	128.54	201.03	296.34	412.84

results of convergence study of a free circular plate with the thickness–radius ratio $\gamma = 0.3$. Table 3 gives the results of convergence study of a clamped circular plate with the thickness–radius ratio $\gamma = 0.2$. Table 4 gives the results of convergence study of a soft simply supported circular plate with the thickness–radius ratio $\gamma = 0.1$. It is shown that convergence is satisfactory. Taking the free circular plate as an example, 10×5 terms of the Chebyshev polynomials can give the first six frequency parameters at least accurate to four significant figures. In general, with the increase of the plate thickness, more terms of the Chebyshev polynomials in the thickness direction and fewer terms in the radial direction are needed.

Table 5 shows the results of a comparison study for a clamped thin circular plate with thickness–radius ratio $\gamma = 0.001$, which is resting on a Winkler foundation with the stiffness parameters $\Phi_1 = 10^{-6}$ and $\Phi_2 = 0$. To be consistent with the presentation of results from two-dimensional analysis, the dimensionless frequency parameter $\lambda = \omega R^2 \sqrt{\rho h / D}$ is used here. The classical solution of a plate resting on Winkler foundation can be directly derived from the classical solutions $\hat{\lambda}$ [1] of the plate without foundation, i.e. $\lambda^2 = \hat{\lambda}^2 + k_1 R^4 / D$. The first five frequency parameters, respectively, for circumferential wavenumber $s = 0, 1, 2, 3$ were presented. Very good agreement is observed.

4. Parametric study

The numerical procedure of the three-dimensional elasticity solution developed in the previous section is applied to extract the vibration frequencies and mode shapes of the plate-foundation system. In the following computations, the Poisson's ratio is taken as $v = 0.3$.

Table 6 gives the first five frequency parameters Ω of axisymmetric vibration ($s = 0$) and circumferential vibrations ($s = 1, 2$) for a thick circular plate with the thickness–radius ratio $\gamma = 0.25$. Two groups of different foundation stiffness parameters, namely (a) $\Phi_1 = 10$ and $\Phi_2 = 1$, and (b) $\Phi_1 = 100$ and $\Phi_2 = 10$, and three kinds of boundary conditions, namely (a) clamped, (b) free, and (c) hard simply supported, are considered. A clamped circular plate with thickness–radius ratio $\gamma = 0.3$ resting on Winkler foundation ($\Phi_2 = 0$) is analyzed. Figs. 2–4 show the effects of foundation stiffness on the first four frequency parameters, respectively, for $s = 0, 1, 2$. From Table 6 and Figs. 2–4, it is observed that with the increase of the foundation stiffness, the frequency parameters monotonically increase. Moreover, it is seen that with the increase in foundation stiffness, the frequency parameters all approach constant values sooner or later according to their order. However, this phenomenon is not observed with the use of two-dimensional approximate theories such as the classical plate theory and the first-order shear deformation theory. Using these simplified methods for analysis, the frequency parameters keep on increasing and therefore infinite foundation stiffness implies infinite frequencies. It is clear that for a thick plate, such a conclusion is obviously wrong. This is because with the increase in plate thickness, a plate with free boundary conditions tends to act as a beam with sliding-free ends. In the two-dimensional theories, the foundation is assumed to be acting on the median surface of the

Table 6
The first five frequency parameters Ω for a thick circular plate ($\gamma = 0.25$) on two different Pasternak foundations

s	Φ_1,Φ_2	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Plate with clamped edge						
0	10,1	6.082	7.001	8.498	10.101	11.965
	100,10	6.330	9.540	10.507	11.093	12.477
1	10,1	3.330	5.366	7.300	8.515	8.777
	100,10	3.331	5.368	8.190	8.603	10.325
2	10,1	5.163	6.848	8.233	9.872	9.965
	100,10	5.168	6.870	9.300	10.010	10.909
Plate with completely free edge						
0	10,1	3.410	6.314	7.242	8.349	9.855
	100,10	3.419	8.044	9.094	10.091	10.705
1	10,1	2.728	5.637	6.719	7.427	8.000
	100,10	2.729	5.778	6.787	9.043	9.264
2	10,1	2.345	4.215	6.866	8.012	8.316
	100,10	2.345	4.219	7.313	8.316	9.261
Plate with hard simply supported edge						
0	10,1	3.416	6.603	7.485	9.619	10.112
	100,10	3.424	8.057	10.145	10.254	11.440
1	10,1	1.162	5.309	5.783	7.808	8.519
	100,10	1.162	5.310	5.911	8.533	9.192
2	10,1	2.425	6.670	7.180	9.449	9.521
	100,10	2.425	6.672	7.660	9.957	10.017

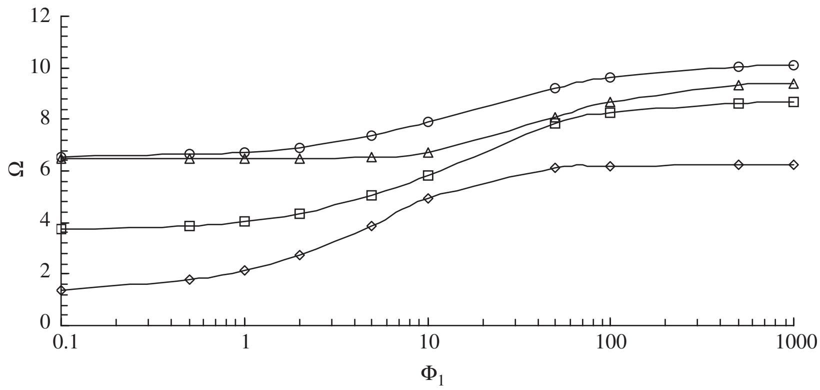

Fig. 2. The first four frequency parameters of axisymmetric vibration for a clamped circular plate with thickness–radius ratio $\gamma = 0.3$ and resting on Winkler foundation.

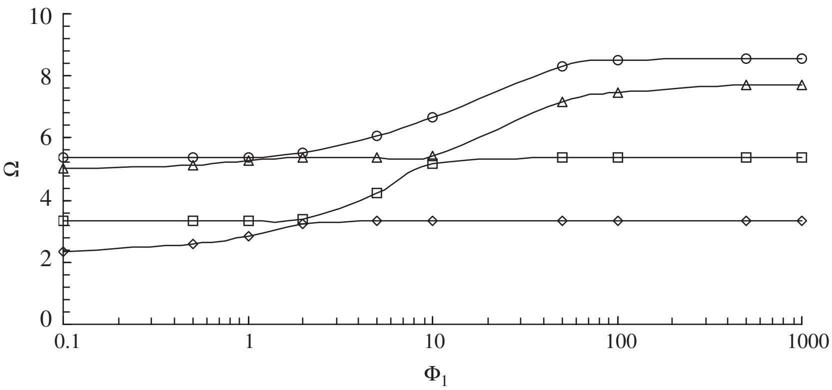

Fig. 3. The first four frequency parameters of circumferential vibration for $s = 1$ for a clamped circular plate with thickness–radius ratio $\gamma = 0.3$ and resting on Winkler foundation.

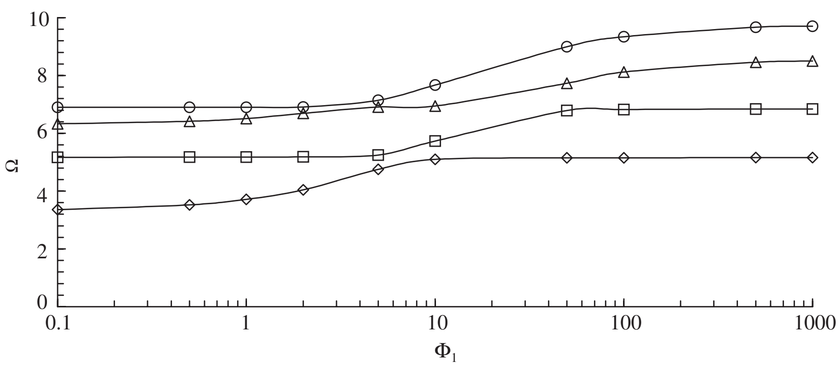

Fig. 4. The first four frequency parameters of circumferential vibration for $s = 2$ for a clamped circular plate with thickness–radius ratio $\gamma = 0.3$ and resting on Winkler foundation.

plate but not on the lower surface. Therefore even for a thin plate, provided that the foundation stiffness is sufficiently large, the two-dimensional theories still give erroneous results. With the increase of the foundation stiffness, the frequencies of the flexural modes increase accordingly. However, the frequencies of the extensional modes, which cannot be analyzed by the thin plate theory or the moderately thick plate theory, vary very slowly. Therefore, with an increase in the foundation stiffness, the frequencies of the extensional modes, which originally fall into the band of the higher-order frequencies, enter into the band of the lower-order frequencies of the plate. Take a clamped thin plate with the thickness–radius ratio $\gamma = 0.01$ resting on Winkler foundation ($\Phi_2 = 0$) as an example. For a foundation having a stiffness parameter $\Phi_1 = 1$, the three-dimensional elasticity theory gives the first frequency parameter of the axisymmetric mode as $\lambda = 1327.6$, but the classical plate theory gives $\lambda = 2049.4$. However, when the stiffness parameter increases to $\Phi_1 = 1000$, the three-dimensional elasticity theory and the classical plate theory give the first frequency parameter of the axisymmetric mode as $\lambda = 1327.9$ and $\lambda = 64807.4$, respectively. This phenomenon can also be observed from Table 6 and Figs. 2–4. Taking Fig. 3 as an example, for a foundation having a stiffness parameter $\Phi_1 = 0.1$, the first and third frequencies of the plate are the flexural modes while the second and fourth frequencies of the plate are the extensional modes. With further increase in foundation stiffness, the frequencies of the flexural modes increase gradually while those of the extensional modes stay very much the same. For $\Phi_1 > 2$, the fundamental frequency of the extensional modes falls below the fundamental frequency of the flexural modes, while for $\Phi_1 > 10$, the second frequency of the extensional modes also falls below the fundamental frequency of the flexural modes. In such a case, the first two frequencies of the plate entirely belong to the extensional modes.

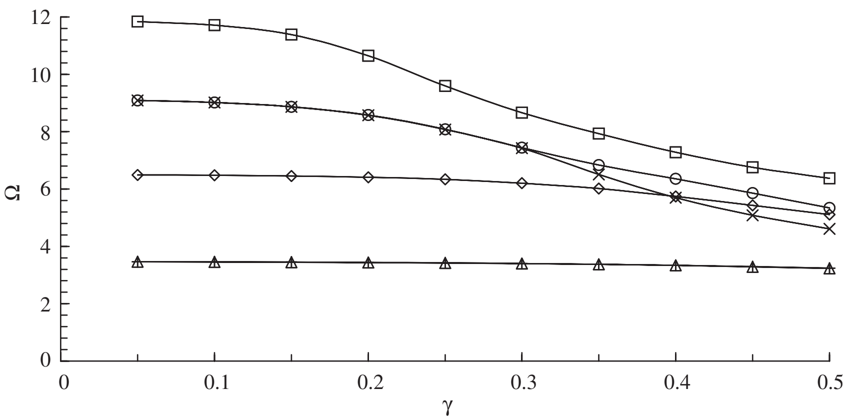

Fig. 5. The first two frequency parameters of axisymmetric vibration: clamped plates: ◇ first mode, □ second mode; completely free plates: △ first mode, × second mode; hard simply supported plates: + first mode, ○ second mode.

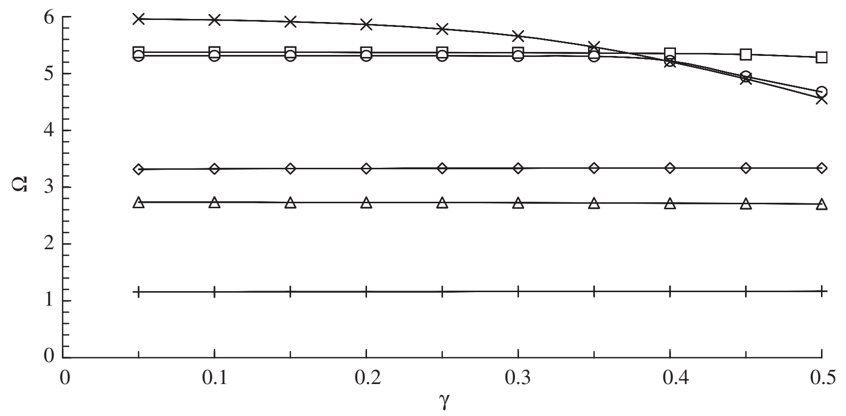

Fig. 6. The first two frequency parameters of circumferential vibration for $s = 1$: clamped plates: ◇ first mode, □ second mode; completely free plates: △ first mode, × second mode; hard simply supported plates: + first mode, ○ second mode.

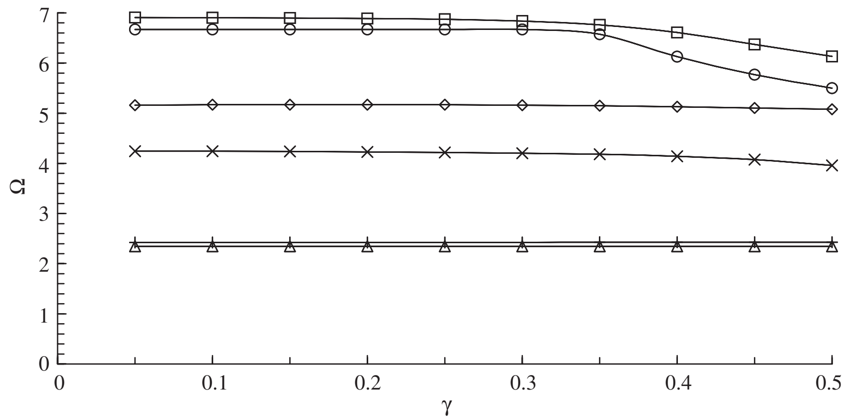

Fig. 7. The first two frequency parameters of circumferential vibration for $s = 2$; clamped plates: ◇ first mode, □ second mode; completely free plates: △ first mode, × second mode; hard simply supported plates: + first mode, ○ second mode.

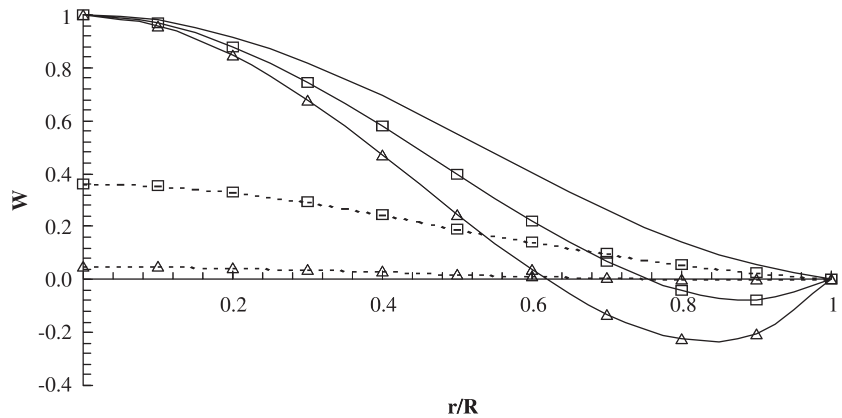

Fig. 8. The fundamental axisymmetric mode shape W at the upper and lower surfaces of a clamped circular plate with thickness–radius ratio $\gamma = 0.25$ and resting on Pasternak foundation: —no foundation; —□— upper surface for $\Phi_1 = 10$ and $\Phi_2 = 1$; ···□··· lower surface for $\Phi_1 = 10$ and $\Phi_2 = 1$; —△— upper surface for $\Phi_1 = 50$ and $\Phi_2 = 5$; ···△··· lower surface for $\Phi_1 = 50$ and $\Phi_2 = 5$.

Figs. 5–7 show the effects of the plate thickness–radius ratio on the first two frequency parameters of the axisymmetric vibrations and circumferential vibrations ($s = 1, 2$), respectively. The Pasternak foundation stiffness parameters are taken as $\Phi_1 = 200$ and $\Phi_2 = 20$. Three kinds of boundary conditions, namely (a) clamped, (b) completely free, and (c) hard simply supported, are considered. It can be seen that with the increase of the thickness–radius ratio, the frequency parameters decrease monotonically. However, the fundamental frequency parameters, except for the axisymmetric vibration of clamped plates, are relatively insensitive to changes in the plate thickness.

Fig. 8 gives the fundamental mode shape W at the upper and lower surfaces for axisymmetric vibration of a clamped circular plate with the thickness–radius ratio $\gamma = 0.25$. Two kinds of foundation stiffness parameters are considered: (a) $\Phi_1 = 10$ and $\Phi_2 = 1$, and (b) $\Phi_1 = 50$ and $\Phi_2 = 5$. These mode shapes are compared with those without foundation when the mode shapes at the upper and lower surfaces should be the same. It is observed that the displacement at the lower surface is always smaller than that at the upper surface, and the difference between the displacements at the lower and upper surfaces increases with the increase of foundation stiffness. For a very stiff foundation, the displacement at the lower surface in the z direction is close to zero.

For plates with free edges, the effect of medium beyond the plate edge on frequency parameters has been evaluated by considering plates with thickness–radius ratios $\gamma = h/R = 0.1$ and $\gamma = h/R = 0.2$. The results for

Table 7
The effect of medium beyond the plate edge on the first six frequency parameters of a free circular plate with thickness-radius ratio $\gamma = 0.1$

Φ_1,Φ_2	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$s = 0$						
10,1	3.456	8.986	10.912	13.841	15.304	17.811
	(3.456)[a]	(8.929)	(9.615)	(13.588)	(14.346)	(17.742)
10,5	3.457	9.007	13.428	14.095	18.317	21.186
	(3.456)	(8.947)	(9.618)	(14.050)	(18.307)	(20.661)
100,10	3.457	9.014	14.042	18.374	21.468	22.853
	(3.457)	(9.013)	(14.036)	(18.345)	(21.276)	(22.233)
100,50	3.457	9.015	14.051	18.412	21.551	23.043
	(3.457)	(9.014)	(14.047)	(18.385)	(21.340)	(22.260)
$s = 1$						
10,1	2.733	5.940	6.834	9.946	11.389	12.752
	(2.733)	(5.938)	(6.833)	(9.945)	(10.735)	(11.569)
10,5	2.733	5.942	6.835	9.946	11.462	13.177
	(2.733)	(5.941)	(6.834)	(9.946)	(11.462)	(13.177)
100,10	2.733	5.943	6.835	9.946	11.474	13.177
	(2.733)	(5.943)	(6.835)	(9.946)	(11.473)	(13.177)
100,50	2.733	5.943	6.835	9.946	11.478	13.177
	(2.733)	(5.943)	(6.835)	(9.946)	(11.477)	(13.177)
$s = 2$						
10,1	2.346	4.241	7.612	8.737	11.406	13.450
	(2.346)	(4.241)	(7.609)	(8.732)	(11.406)	(12.529)
10,5	2.346	4.241	7.616	8.747	11.408	13.605
	(2.346)	(4.241)	(7.615)	(8.746)	(11.408)	(13.589)
100,10	2.346	4.241	7.617	8.749	11.409	13.623
	(2.346)	(4.241)	(7.617)	(8.748)	(11.409)	(13.620)
100,50	2.346	4.241	7.617	8.749	11.409	13.631
	(2.346)	(4.241)	(7.617)	(8.749)	(11.409)	(13.630)

[a] *Note*: Entries in parentheses are those when the effect of medium beyond the plate edge is ignored.

four groups of foundation stiffness parameters are compared with those obtained by ignoring the effect of medium beyond the plate edge. The first six frequency parameters for axisymmetric vibration ($s = 0$) and circumferential vibrations ($s = 1, 2$) are given in Tables 7 and 8, respectively. It is observed that when the effect of medium beyond the plate edge is taken into account, the natural frequencies of the plate-foundation system are always higher than those when the effect is ignored. For most modes, the influence of the medium beyond the plate edge on vibration characteristics of the plate-foundation system is insignificant. However, for some special modes, the effect of medium beyond the plate edge on frequencies is significant. For examples, for foundation stiffness parameters $\Phi_1 = 10$ and $\Phi_2 = 5$, the frequencies of the third axisymmetric mode for a plate with thickness–radius ratio $\gamma = 0.1$ and the second axisymmetric mode for a plate with thickness–radius ratio $\gamma = 0.2$ are at least 25% higher than those when the effect of medium beyond the plate edge is ignored. This means that for these modes, the potential energies provided by the generalized shear forces are significant and cannot be ignored in the analysis.

5. Conclusions

An accurate method for three-dimensional vibration analysis of thick circular plates resting on Pasternak foundation has been presented. By using the Chebyshev polynomial series multiplied by a boundary function as the admissible functions, which satisfies the geometric boundary conditions a priori, an accurate governing frequency equation is derived by the energy functional of the plate-foundation system using the Ritz method. For plates with free edges, the effect of medium beyond the plate edge has been considered by introducing the generalized shear force concept. Convergence and comparison studies demonstrate the accuracy and

Table 8
The effect of medium beyond the plate edge on the first six frequency parameters of a free circular plate with thickness-radius ratio $\gamma = 0.2$

Φ_1,Φ_2	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$s = 0$						
10,1	3.432	7.252	8.134	9.879	10.661	11.125
	(3.430)[a]	(6.447)	(8.051)	(9.210)	(10.656)	(11.056)
10,5	3.434	8.356	8.895	11.043	11.558	12.516
	(3.431)	(6.449)	(8.599)	(10.881)	(11.527)	(12.362)
100,10	3.436	8.560	11.336	11.508	12.844	13.282
	(3.436)	(8.551)	(11.247)	(11.502)	(12.547)	(13.242)
100,50	3.436	8.579	11.447	11.578	13.209	13.581
	3.436	8.569	11.384	11.559	12.579	13.522
$s = 1$						
10,1	2.731	5.810	6.791	8.359	9.340	9.991
	(2.731)	(5.773)	(6.771)	(7.351)	(9.232)	(9.986)
10,5	2.731	5.849	6.808	9.848	10.103	10.929
	(2.731)	(5.837)	(6.805)	(9.090)	(9.921)	(10.465)
100,10	2.731	5.861	6.811	9.894	10.300	11.424
	(2.731)	(5.860)	(6.811)	(9.889)	(10.278)	(11.367)
100,50	2.731	5.863	6.812	9.901	10.348	11.520
	(2.731)	(5.862)	(6.812)	(9.899)	(10.338)	(11.505)
$s = 2$						
10,1	2.346	4.228	7.361	8.332	9.664	10.236
	(2.346)	(4.227)	(7.212)	(8.142)	(8.834)	(10.226)
10,5	2.346	4.229	7.476	8.522	10.890	11.387
	(2.346)	(4.229)	(7.461)	(8.507)	(10.544)	(11.328)
100,10	2.346	4.230	7.494	8.548	11.051	11.487
	(2.346)	(4.229)	(7.492)	(8.547)	(11.008)	(11.487)
100,50	2.346	4.230	7.499	8.557	11.105	11.534
	(2.346)	(4.230)	(7.498)	(8.556)	(11.096)	(11.533)

[a] *Note*: Entries in parentheses are those when the effect of medium beyond the plate edge is ignored.

numerical stability of the present method. The effects of various parameters such as the thickness–radius ratio and the foundation stiffness parameters on natural frequencies of the plate-foundation system are studied in detail. Some results known for the first time are given in tabular and graphical forms. It is shown that for the vibration analysis of plates resting on foundation, the validity of two-dimensional approximate theories such as the classical plate theory and the first-order shear deformation plate theory not only depends on the thickness of the plate but also the foundation stiffness.

Acknowledgment

The work described in this paper was partially supported by a grant "Analysis of building structures using hybrid stress hexahedral elements" from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No: HKU7071/99E).

Appendix

The column vectors $\{A\}$, $\{B\}$ and $\{C\}$ are composed of unknown coefficients as follows:

$$\begin{aligned} \{A\} &= \{ A_{11} \quad \cdots \quad A_{1J} \quad A_{21} \quad \cdots \quad A_{2J} \quad \cdots \quad A_{I1} \quad \cdots \quad A_{IJ} \}^{\mathrm{T}}, \\ \{B\} &= \{ B_{11} \quad \cdots \quad B_{1L} \quad B_{21} \quad \cdots \quad B_{2L} \quad \cdots \quad B_{K1} \quad \cdots \quad B_{KL} \}^{\mathrm{T}}, \\ \{C\} &= \{ C_{11} \quad \cdots \quad C_{1N} \quad C_{21} \quad \cdots \quad C_{2N} \quad \cdots \quad C_{M1} \quad \cdots \quad C_{MN} \}^{\mathrm{T}}. \end{aligned} \quad (24)$$

The elements of the sub-matrices $[K^{ij}]$ and $[M^{ij}]$ $(i,j = u, v, w)$ are given by

$$[K^{uu}] = \frac{1-v}{1-2v}(D^{111}_{ui u\bar{i}} + D^{00-1}_{uiu\bar{i}})H^{00}_{uju\bar{j}} + \frac{v}{1-2v}(D^{010}_{uiu\bar{i}} + D^{100}_{uiu\bar{i}})H^{00}_{uju\bar{j}} + \frac{1}{2\gamma^2}(s^2\gamma^2 D^{00-1}_{uiu\bar{i}}H^{00}_{uju\bar{j}} + D^{001}_{uiu\bar{i}}H^{11}_{uju\bar{j}}),$$

$$[K^{uv}] = \frac{(1-v)s}{1-2v}D^{00-1}_{uiv\bar{k}}H^{00}_{ujv\bar{l}} + \frac{vs}{1-2v}D^{100}_{uiv\bar{k}}H^{00}_{ujv\bar{l}} + \frac{s}{2}(D^{00-1}_{uiv\bar{k}} - D^{010}_{uiv\bar{k}})H^{00}_{ujv\bar{l}},$$

$$[K^{uw}] = \frac{v}{(1-2v)\gamma}(D^{101}_{uiw\bar{m}} + D^{000}_{uiw\bar{m}})H^{01}_{ujw\bar{n}} + \frac{1}{2\gamma}D^{011}_{uiw\bar{m}}H^{10}_{ujw\bar{n}},$$

$$[K^{vv}] = \frac{(1-v)s^2}{1-2v}D^{00-1}_{vkv\bar{k}}H^{00}_{vlv\bar{l}} + \frac{1}{2}(D^{111}_{vkv\bar{k}} + D^{00-1}_{vkv\bar{k}} - D^{010}_{vkv\bar{k}} - D^{100}_{vkv\bar{k}})H^{00}_{vlv\bar{l}} + \frac{1}{2\gamma^2}D^{001}_{vkv\bar{k}}H^{11}_{vlv\bar{l}},$$

$$[K^{vw}] = \frac{vs}{(1-2v)\gamma}D^{000}_{vkw\bar{m}}H^{01}_{vlw\bar{n}} + \frac{s}{2\gamma}D^{000}_{vkw\bar{m}}H^{10}_{vlw\bar{n}};$$

$$[K^{ww}] = \frac{1-v}{(1-2v)\gamma^2}D^{001}_{wmw\bar{m}}H^{11}_{wnw\bar{n}} + \frac{1}{2}(D^{111}_{wmw\bar{m}} + s^2 D^{00-1}_{wmw\bar{m}})H^{00}_{wnw\bar{n}} + (-1)^{(n+\bar{n})}\left(\frac{1}{4}\Phi_1 D^{001}_{wmw\bar{m}} + \Phi_2 D^{111}_{wmw\bar{m}} + \Phi_2 s^2 D^{00-1}_{wmw\bar{m}}\right)\Big/\gamma$$

for a plate having zero vertical displacements at $\bar{r} = 1$, $\bar{z} = -1$,

$$[K^{ww}] = \frac{1-v}{(1-2v)\gamma^2}D^{001}_{wmw\bar{m}}H^{11}_{wnw\bar{n}} + \frac{1}{2}(D^{111}_{wmw\bar{m}} + s^2 D^{00-1}_{wmw\bar{m}})H^{00}_{wnw\bar{n}} + (-1)^{(n+\bar{n})}\left(\frac{1}{4}\Phi_1 D^{001}_{wmw\bar{m}} + \Phi_2 D^{111}_{wmw\bar{m}} + \Phi_2 s^2 D^{00-1}_{wmw\bar{m}}\right)\Big/\gamma + (-1)^{(n+\bar{n})}\Phi_3$$

for a plate having non-zero vertical displacements at $\bar{r} = 1$, $\bar{z} = -1$,

$$[M^{uu}] = D^{001}_{uiu\bar{i}}H^{00}_{uju\bar{j}}/8, \quad [M^{vv}] = D^{001}_{vkv\bar{k}}H^{00}_{vlv\bar{l}}/8, \quad [M^{ww}] = D^{001}_{wmw\bar{m}}H^{00}_{wnw\bar{n}}/8 \tag{25}$$

in which

$$D^{abc}_{\alpha\sigma\beta\tau} = \int_{-1}^{1} \frac{\mathrm{d}^a[F_\alpha(\bar{r})P_\sigma(\bar{r})]}{\mathrm{d}\bar{r}^a}\frac{\mathrm{d}^b[F_\beta(\bar{r})P_\tau(\bar{r})]}{\mathrm{d}\bar{r}^b}(\bar{r}+1)^c\,\mathrm{d}\bar{r},$$
$$H^{ab}_{\alpha\sigma\beta\tau} = \int_{-1}^{1} \frac{\mathrm{d}^a P_\sigma(\bar{z})}{\mathrm{d}\bar{z}^a}\frac{\mathrm{d}^b P_\tau(\bar{z})}{\mathrm{d}\bar{z}^b}\,\mathrm{d}\bar{z}, \quad a;b = 0;1, \quad c = 0;1;-1,$$
$$\alpha;\beta = u;v;w, \quad \sigma = i;k;m;j;l;n, \quad \tau = \bar{i};\bar{k};\bar{m};\bar{j};\bar{l};\bar{n}. \tag{26}$$

References

[1] A.W. Leissa, *Vibration of Plates (NASA SP-160)*, Office of Technology Utilization, Washington, DC, 1969.

[2] P.L. Pasternak, On a new method of analysis of an elastic foundation by means of two foundation constants, *Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu i Arkhitekture*, Moscow, USSR, 1954, pp. 1–56 (in Russian).

[3] E. Winkler, Die Lehre von der Elasticitaet und Festigkeit, Prag. Dominicus, 1867.

[4] K. Kamal, S. Durvasula, Bending of circular plate on elastic-foundation, *Journal of Engineering Mechanics-ASCE* 109 (1983) 1293–1298.

[5] X.J. Zheng, Y.H. Zhou, Exact solution of nonlinear circular plate on elastic-foundation, *Journal of Engineering Mechanics-ASCE* 114 (1988) 1303–1316.

[6] Z. Celep, Circular plate on tensionless Winkler foundation, *Journal of Engineering Mechanics-ASCE* 114 (1988) 1723–1739.

[7] A.A. Khathlan, Large-deformation analysis of plates on unilateral elastic-foundation, *Journal of Engineering Mechanics* 120 (1994) 1820–1827.
[8] M. Nassar, Bending of a circular plate on a linear viscoelastic foundation, *Applied Mathematical Modelling* 5 (1981) 60–62.
[9] I. Elishakoff, J. Tang, Buckling of polar orthotropic circular plates on elastic-foundation by computerized symbolic algebra, *Computer Methods in Applied Mechanics and Engineering* 68 (1988) 229–247.
[10] G.Z. Voyiadjis, P.I. Kattan, Effect of transverse normal strain on the bending of thick circular plates on an elastic-foundation, *International Journal of Mechanical Sciences* 33 (1991) 413–433.
[11] A.K. Ghosh, Axisymmetric dynamic response of a circular plate on an elastic foundation, *Journal of Sound and Vibration* 205 (1997) 112–120.
[12] C.M. Wang, Y. Xiang, Q. Wang, Axisymmetric buckling of Reddy circular plates on Pasternak foundation, *Journal of Engineering Mechanics-ASCE* 127 (2001) 254–259.
[13] M.H. Omurtag, A. Ozutok, A.Y. Akoz, Free vibration analysis of Kirchhoff plates resting on elastic foundation by mixed finite element formulation based on Gateaux differential, *International Journal for Numerical Methods in Engineering* 40 (1997) 295–317.
[14] N. Eratli, A.Y. Akoz, Free vibration analysis of Reissner plates by mixed finite element, *Structural Engineering and Mechanics* 13 (2002) 277–298.
[15] R. Buczkowski, W. Torbacki, Finite element modelling of thick plates on two-parameter elastic foundation, *International Journal for Numerical and Analytical Methods in Geomechanics* 25 (2001) 1409–1427.
[16] P.C. Dumir, Circular plates on Pasternak elastic foundations, *International Journal for Numerical and Analytical Methods in Geomechanics* 11 (1987) 51–60.
[17] Z. Celep, D. Turhan, Axisymmetrical vibrations of circular plates on tensionless elastic foundation, *Journal of Applied Mechanics-ASME* 57 (1990) 677–681.
[18] J.R. Hutchinson, Axisymmetric flexural vibrations of a thick free circular plates, *Journal of Applied Mechanics-ASME* 46 (1979) 139–144.
[19] J.R. Hutchinson, Vibration of thick free circular plates, exact versus approximate solutions, *Journal of Applied Mechanics-ASME* 51 (1984) 581–585.
[20] J.R. Fan, J.Q. Ye, Exact-solutions for axisymmetric vibration of laminated circular plates, *Journal of Engineering Mechanics-ASCE* 116 (1990) 920–927.
[21] H.J. Ding, R.Q. Xu, Exact solutions for free vibrations of transversely isotropic circular plates, *Acta Mechanica Solida Sinica* 13 (2000) 105–111.
[22] K.M. Liew, B. Yang, Three-dimensional elasticity solutions for free vibrations of circular plates: a polynomials-Ritz analysis, *Computer Methods in Applied Mechanics and Engineering* 175 (1999) 189–201.
[23] K.M. Liew, B. Yang, Elasticity solutions for free vibrations of annular plates from three-dimensional analysis, *International Journal of Solids and Structures* 37 (2000) 7689–7702.
[24] J. So, A.W. Leissa, Three-dimensional vibrations of thick circular and annular plates, *Journal of Sound and Vibration* 209 (1998) 15–41.
[25] D. Zhou, Y.K. Cheung, S.H. Lo, F.T.K. Au, Three-dimensional vibration analysis of rectangular thick plates on Pasternak foundation, *International Journal for Numerical Methods in Engineering* 59 (2004) 1313–1334.
[26] D. Zhou, F.T.K. Au, Y.K. Cheung, S.H. Lo, Three-dimensional vibration analysis of circular and annular plates via the Chebyshev-Ritz method, *International Journal of Solids and Structures* 40 (2003) 3089–3105.
[27] L. Fox, I.B. Parker, *Chebyshev Polynomials in Numerical Analysis*, Oxford University Press, London, 1968.
[28] A.P.S. Selvadurai, *Elastic Analysis of Soil-foundation Interaction*, Amsterdam, Elsevier, 1979.